TENSOR FIELDS ON MANIFOLDS

def: A C^∞- covariant tensor field of order r on a C^∞- manifold is a function

$$\Phi : M \ni p \longmapsto \phi_p \in \mathcal{T}^r(T_p M)$$

such that $\forall$ $X_1, \dots, X_r$ C^∞- vector fields on an open subset $\mathcal{U} \subset M$ then

$$\Phi(X_1, \dots, X_r) : \mathcal{U} \longrightarrow \mathbb{R}$$ is a C^∞- function

$$\mathcal{T}^r(M) := \left\{ \begin{array}{l} C^\infty \text{ covariant vector fields of order } r \\ \text{on } M \end{array} \right\}$$

Examples:

- $r = 1$: covector
- $r = 2$: bilinear forms

Rmk: A covariant tensor field of order r $\phi \in \mathcal{T}^r(M)$ is $C^\infty(M)$ - linear in each variable:

If $f \in C^\infty(M)$:

$$\phi(X_1, \dots, f X_i, \dots X_r) = f \Phi(X_1, \dots, X_i, \dots, X_r)$$

because Φ_p ~~[illegible]~~ $\mathbb{R}$-linear $\forall p \in M$.

Recall: V vector space $\rightsquigarrow$ $\Phi : V \times V \times \dots \times V \longrightarrow \mathbb{R}$ is determined by its components

If (U, φ) coordinate neighborhood &

$E_1, \dots, E_n$ coordinate frames

$$\Rightarrow \underbrace{\Phi(E_{j_1}, \dots, E_{j_r})}_{\text{components of } \Phi} \text{ determine } \Phi$$

FACTS / Exercises:

① $\mathcal{T}^r(M)$ is a vector space over $\mathbb{R}$

② let $F: M \to N$ ~~be~~ C^∞-map of manifolds. Then each C^∞-covariant tensor field Φ on N determines a C^∞-covariant tensor field $F^*\Phi$ on M by the formula:

$$(F^*\Phi)_p(X_{1p}, \dots, X_{rp}) = \Phi_p(F_*(X_{1p}), \dots, F_*(X_{rp}))$$

The map $F^*: \mathcal{T}^r(N) \longrightarrow \mathcal{T}^r(M)$ is linear & takes $\left|\begin{array}{l}\text{symmetric}\\ \text{alternating}\end{array}\right.$ tensors to $\left|\begin{array}{l}\text{symmetric}\\ \text{alternating}\end{array}\right.$ tensors.

③ $\mathcal{A}: \mathcal{T}^r(M) \longrightarrow \mathcal{T}^r(M)$
$\mathcal{S}: \mathcal{T}^r(M) \longrightarrow \mathcal{T}^r(M)$ are defined in the same way as $M = V$

Similarly,

(1) $\mathcal{A}^2 = \mathcal{A}$; $\mathcal{J}^2 = \mathcal{J}$

(2) $\mathcal{A}(\mathcal{T}^r(M)) = \Lambda^r(M)$ & $\mathcal{J}(\mathcal{T}^r(M)) = \Sigma^r(M)$

(3) ϕ alternating $\Leftrightarrow \mathcal{A}\phi = \phi$
symmetric $\Leftrightarrow \mathcal{J}\phi = \phi$

(4*) If $F: M \to N$ is C^∞-mapping
$\Rightarrow \mathcal{A}$ and $\mathcal{J}$ commute wt $F^*: \mathcal{T}^r(N) \to \mathcal{T}^r(M)$

multiplication of tensor fields

can be defined as well:

If $\varphi \in \mathcal{T}^r(M)$
$\psi \in \mathcal{T}^s(M)$ then $\varphi \otimes \psi$

$(\varphi \otimes \psi)_p := \varphi_p \otimes \psi_p \in \mathcal{T}^{r+s}(T_pM)$

defines a tensor field $\Leftrightarrow C^\infty$!

we need to check in local coordinates :

$$(\varphi \otimes \psi)(E_{i_1}, \dots, E_{i_{r+s}}) = \underbrace{\underbrace{\varphi(E_{i_1}, \dots, E_{i_r})}_{C^\infty} \cdot \underbrace{\psi(E_{i_{r+1}}, \dots, E_{i_{r+s}})}_{C^\infty}}_{\text{product of } C^\infty\text{- functions}}$$

$\Downarrow$

C^∞ !

Thm: The mapping
$\mathcal{T}^r(M) \times \mathcal{T}^s(M) \longrightarrow \mathcal{T}^{r+s}(M)$ bilinear & associative.

If $\omega^1, \dots, \omega^n$ is a basis of $\mathcal{T}^1(M)$ then every element of $\mathcal{T}^r(M)$ is a linear combination wt C^∞- coefficients of $\{\omega^{i_1}\otimes\cdots\otimes\omega^{i_r}\}_{1\le i_1,\dots,i_r\le n}$

If $F: N\to M$ is a C^∞-mapping, $\varphi\in\mathcal{T}^r(M)$, $\psi\in\mathcal{T}^s(M)$

$$\Rightarrow F^*(\varphi\otimes\psi) = F^*(\varphi)\otimes F^*(\psi) \in \mathcal{T}^r(N)$$

proof: at each point.
+
result when $M = V$ $\square$

$E_1,\dots,E_n$ coordinate frame $\quad E_i = \theta_*^{-1}\left(\frac{\partial}{\partial x^i}\right)$

$\omega^1,\dots,\omega^n$ duals $\quad \omega^i = \theta^*(dx^i)$ $\quad (*)$

Corollary: $(U,\theta)\subset M$ coordinate neighborhood

Each $\varphi\in\mathcal{T}^r(U)$ has a unique expression of the form

$$\varphi = \sum_{i_1}\cdots\sum_{i_r} a_{i_1\dots i_r}\,\omega^{i_1}\otimes\cdots\otimes\omega^{i_r}$$

where at each point of U,

$$a_{i_1\dots i_r} = \varphi(E_{i_1},\dots,E_{i_r})$$ are the components of φ in the basis $\{\omega^{i_1}\otimes\cdots\otimes\omega^{i_r}\}$ are $C^\infty(U)$.

~~proof: apply thm here~~

IMPORTANT

EXAMPLE M manifold $\theta \in \Lambda^2(M)$

& $U \subset M$ open subset $i: U \longrightarrow M$ inclusion

$\Rightarrow$ $i^*: \Lambda^2 M \longrightarrow \Lambda^2 U$

$\theta \longmapsto i^*\theta =: \theta_{|U}$

(U, φ) coordinate neighborhood

$\varphi(q) = (x^1(q), \dots, x^n(q))$ coordinate functions on U

$\Downarrow$

$dx^1, \dots, dx^n$ linearly indipendent elements of $\Lambda^1 U$

& C^∞-field of coframes on U

$\Lambda(U)$ generated by 1, $\langle dx^1, \dots, dx^n \rangle$ as algebra over $C^\infty(U)$

- $\Lambda^0(U) = C^\infty(U)$
- $\Lambda^1(U) = \langle dx^1, \dots, dx^n \rangle$

$\Rightarrow$ <u>locally</u> every k-form θ on M has a unique repres. on U of the form:

$$\theta_U = \sum_{1 \le i_1 < \dots < i_k \le n} a_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

wt $a_{i_1 \dots i_k} \in C^\infty(U)$

$$= \sum_{\substack{\text{all} \\ i_1, \dots, i_k \\ \text{(repetition included)}}} \frac{1}{k!} b_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$U \subset M$ open set

$E_1, \dots, E_n$ field of frames

$\wr$ dual basis

$\omega^1, \dots, \omega^n$ $\quad \omega^i(E_j) = \delta^i_j$ <u>coframes</u>

$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$

$\wr$

$dx^1 \dots dx^n$

(U, φ) $\quad \varphi(q) = (x^1(q), \dots, x^n(q))$

The exterior algebra on manifolds

def: An alternating covariant tensor field of order r on M will be called an exterior differential form of degree r (r-form)

$$\Lambda^r(M) := \{ r\text{-forms on } M \} \underset{\text{subspace}}{<} \mathcal{T}^r(M)$$

Thm:

let $\Lambda(M)$ denote the vector space over $\mathbb{R}$ of all exterior differential forms. Then for

$\varphi \in \Lambda^r(M)$ the formula

$\psi \in \Lambda^s(M)$

$(\varphi \wedge \psi)_p := \varphi_p \wedge \psi_p$ defines an associative product

wt $(\varphi \wedge \psi) = (-1)^{rs} \cdot \psi \wedge \varphi$

Furthermore, (1) $(\Lambda(M), \wedge)$ is an algebra over $\mathbb{R}$.

(2) If $f \in C^\infty(M)$ then $(f\varphi) \wedge \psi = f(\varphi \wedge \psi)$
$= \varphi \wedge (f\psi)$.

(3) If $\omega^1, \dots, \omega^n$ is a field of coframes on M

$\Rightarrow \{ \omega^{i_1} \wedge \dots \wedge \omega^{i_r} \}_{1 \le i_1 < \dots < i_r \le n}$ is a basis of $\Lambda^r(M)$.

Theorem:

If $F: M \to N$ is a C^∞-mapping of manifolds

$\Rightarrow F^*: \Lambda(N) \to \Lambda(M)$ is an algebra homomorphism.

$\Lambda(M)$ = exterior algebra on M / algebra of differential forms on M

ORIENTED VECTOR SPACE

V vector space $\dim V = n$

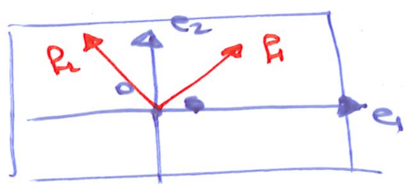
def: Two bases $\{e_1, \dots, e_n\}$ & $\{f_1, \dots, f_n\}$ have the same orientation if $\det(\alpha_i^j) > 0$ where $f_i = \sum_{j=1}^{n} \alpha_i^j e_j$

Rmk: This is an equivalence relation on $\{$all bases of $V\}$ wt 2 equivalence classes

def: An orientation on V is the choice of one equivalence classes $\{e_1, \dots, e_n\}$. All basis wt the same orientation of V are positively oriented.

$(V, [\{e_1, \dots, e_n\}])$ oriented vector spaces

EX. $V = \mathbb{R}^2$ $\{e_1, e_2\}$ canonical basis
$e_1(1,0)$ $e_2(0,1)$



$\{f_1 = (1,1) ; f_2 = (-1,1)\}$ same orientation

$\{\mathbb{R}^2, \{(-1,0) ; (0,1)\}\}$ opposite orientation

We can express the same concept using $\Lambda^n V$

Recall! $\dim \Lambda^n V = \binom{n}{n} = 1$

$\Rightarrow$ every non-zero n-form is a basis for $\Lambda^n V$

lemma:

Let $\Omega \neq 0$ be an alternating covariant tensor on V of order $n = \dim V$. Let $e_1, .., e_n$ be a basis of V. Then for any set of vectors $v_1, .., v_n$ wt

$v_i = \sum \gamma_i^j e_j$ we have:

$$\Omega(v_1, .., v_n) = \det(\gamma_j^i)\, \Omega(e_1, .., e_n)$$

Expl.: up to a non-zero scalar $(\Omega(e_1, .., e_n))$ Ω coincides wt the determinant of the components of its variables

proof:

$$\Omega(v_1, .., v_n) \overset{\text{multilinear}}{=} \sum_{j_1 \dots j_n} \alpha_1^{j_1} \dots \alpha_n^{j_n}\, \Omega(e_{j_1}, \dots, e_{j_n})$$

$$\overset{\text{antisymm.}}{=} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \cdot \alpha_1^{\sigma(1)} \dots \alpha_n^{\sigma(n)}\, \Omega(e_1, \dots, e_n)$$

$$\underset{\text{definition of determinant}}{=} \det(\alpha_i^j) \cdot \Omega(e_1, \dots, e_n) \quad \boxtimes$$

Corollary: A non-vanishing $\Omega \in \Lambda^n V$ has the same sign on two bases iff they have the same orientation.

proof:

Use the formula above $\boxtimes$

Fixing an orientation on V $\Leftrightarrow$ choosing $\Omega \neq 0$ on V

Corollary:

Two forms Ω_1 and $\Omega_2 \in \Lambda^n V \setminus \{0\}$ determine the same orientation on V iff

$$\Omega_1 = \lambda \cdot \Omega_2 \quad \text{wt } \lambda \in \mathbb{R}^+$$

proof: ~~The forms have the same orientation iff they have the same sign~~

$\dim \Lambda^n V = 1$

If $\Omega_1, \Omega_2 \neq 0 \Rightarrow \exists \lambda \in \mathbb{R}^* \quad \Omega_1 = \lambda \cdot \Omega_2$

$\{v_1, \dots, v_n\}$ basis $\quad \operatorname{sgn}(\Omega_1(v_1, \dots, v_n)) = \lambda \cdot \operatorname{sgn}(\Omega_2(v_1, \dots, v_n))$

$=$ iff $\lambda > 0$

(Two forms determine the same orientation $\Leftrightarrow$ they have the same sign on all bases)

Example :

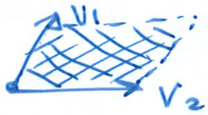
U vector space wt $\Phi(v, w)$ positive definite inner product

- Choose an orthonormal basis $\{e_1, .., e_n\}$ to determine its orientation.
- Choose Ω n-form wt $\Omega(e_1, .., e_n) = +1$

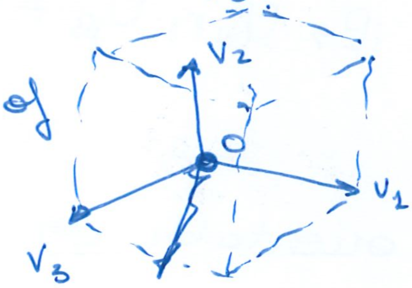
If $\{f_1, .., f_n\}$ is another orthonormal basis

$$\Rightarrow \quad \Omega(f_1, .., f_n) = \det(\alpha_i^j) \, \Omega(e_1, .., e_n)$$

$$= \pm 1$$ (depending on orientation wt respect to $\{e_1, .., e_n\}$)



$\underline{n=2}$: $\Omega(v_1, v_2)$ = area of



$\underline{n=3}$: $\Omega(v_1, v_2, v_3)$ = volume of

def: M manifold is orientable if it is possible to define a C^∞ n-form Ω on M which is not zero at any point. If such a form exists

$\Rightarrow$ M is oriented by the choice of Ω

~~Rmk~~ Example :

$(\mathbb{R}^n, \tilde{\Omega} = dx^1 \wedge \dots \wedge dx^n)$ natural orientation of $\mathbb{R}^n$

def : Let (M_1, Ω_1) oriented manifolds.
(M_2, Ω_2)

$F : M_1 \to M_2$ orientation preserving if

$F^* \Omega_2 = \lambda \cdot \Omega_1$ wt $\lambda : M \to \mathbb{R}^+$ positive C^∞ function

def 2 : M orientable if it can be covered wt Coherently oriented coordinate neighborhoods $\{U_\alpha, \varphi_\alpha\}$
that is, if $U_\alpha \cap U_\beta \neq \emptyset \Rightarrow \varphi_\alpha \circ \varphi_\beta^{-1}$ orientation-preserving

Thm : M orientable $\Leftrightarrow$ def 2

proof :
Assume M orientable and choose $\Omega \in \Lambda^n M$ wt $\Omega \neq 0$ ~~determ~~

Choose $\{U_\alpha, \varphi_\alpha\}$ covering of M ; wt local coordinates $x_\alpha^1, \dots, x_\alpha^n$ such that $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset (M, \Omega)$

$(\varphi_\alpha^{-1})^* \Omega_{|U_\alpha} = \underbrace{\lambda_\alpha(x)}_{\neq 0} dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n$ wt $\lambda_\alpha > 0$

Using previous lemma & corollary : if $U_\alpha \cap U_\beta \neq \emptyset$:

$0 < \lambda_\alpha \det\left(\frac{\partial x_\alpha^i}{\partial x_\beta^j}\right) = \lambda_\beta > 0$ by hypothesis

$\Longrightarrow \det\left(\frac{\partial x_\alpha^i}{\partial x_\beta^j}\right) > 0 \Rightarrow$ charts are coherently oriented .